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1985 J. Phys. A: Math. Gen. 18 467

(http://iopscience.iop.org/0305-4470/18/3/022)

J. Phys. A: Math. Gen. 18 (1985) 467-477. Printed in Great Britain

The quasi-classical propagator of a quantum particle in a uniform field in a half space

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Received 7 August 1984

Abstract. The quasi-classical formulae for the propagator of the Schrödinger equation and for the equilibrium density matrix are obtained for a quantum particle moving in a uniform force field in a half space over an ideal reflecting wall.

1. Introduction

We consider the motion of a quantum particle in a uniform field with the potential $U(x) = -Fx$ in the half space $x > 0$, bounded by an impervious ideal reflecting wall. Solutions of the Schrödinger equation $(\hat{H} - E)\psi = 0$ for this problem are given, for example, in (Flügge 1971). Green functions of this equation are also known (Lukes and Somaratna 1969, Moyer 1973, Tachibana *et al* 1977). The propagator of the time-dependent Schrödinger equation in the case of the motion in the unbounded region $-\infty < x < \infty$ was obtained by Kennard (1927):

$$K(x_2, x_1, t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left[\frac{i}{\hbar}\left(\frac{m(x_2 - x_1)^2}{2t} + \frac{Ft(x_1 + x_2)}{2} - \frac{F^2t^3}{24m}\right)\right]. \tag{1}$$

Here x_1 is the initial point, and x_2 is the final point (the initial moment of time is assumed to be $t = 0$).

In the presence of a rigid wall the propagator is much more complicated, and the best result is an integral representation of K derived by Moyer (1973). Formally, the propagator can be written in the form of series

$$K(x_2, x_1, t) = \sum_n \psi_n(x_2, t)\psi_n^*(x_1, 0) \tag{2}$$

over the complete set of solutions ψ_n of the Schrödinger equation. Nonetheless, although the functions $\psi_n(x, t)$ are known, the direct summation in (2) can hardly be performed explicitly, because of the complicated dependence of these functions on the index n (the energy eigenvalues are determined by the transcendental equation). The aim of our paper is to obtain an approximate expression for the propagator in the presence of a wall in the frame of the quasi-classical approach. Our interest in this problem is explained by several reasons.

Firstly, it is well known that the exact propagator in the case of the motion in the full space coincides with the semiclassical propagator, because the Hamiltonian $\hat{H} = p^2/2m - Fx$ belongs to the class of the so-called quadratic Hamiltonians, if $-\infty < x < \infty$

(see § 2). On the other hand, if $F = 0$, then both the full space propagator and the half space propagator can be calculated exactly via the quasi-classical formulae (see § 3).

Therefore one can ask a natural question, whether the quasi-classical approach is effective in the case of $F \neq 0$. We show that the answer is affirmative in the weak force limit and is negative in the strong force case. In other words, one of our goals is to investigate the applicability of the quasi-classical method of calculating the propagators of non-quadratic systems, considering as an example the simplest non-quadratic ('nearly quadratic') system. In this context our paper is close to the paper by Crandall (1983) who studied the problem of the representation of the propagator in the form of a countable sum of quasi-classical terms. He showed that for the free motion in the two-dimensional region bounded by the rigid walls $\varphi = 0$ and $\varphi = \alpha$ (in polar coordinates) such a representation (a 'collapse' of the propagator) takes place only in exceptional cases, when $\alpha = \pi/b$, b being an integer. Our results show that for the forced motion in one dimension in the presence of a wall the propagator approximately 'collapses' in the weak force limit and does not 'collapse' when the force is not small.

The problem under study is also interesting from several physical points of view. For example, it can be considered as the problem of calculating the S state radial propagator of a particle moving in the three-dimensional spherically symmetric potential $V(r) = F|r|$ which describes in the lowest approximation the interaction between two quarks. Besides, in recent years the problems relating to the motion of quantum particles in the semiaxis $0 < x < \infty$ have attracted the attention of specialists in quantum field theory because such quantum mechanical systems can serve as the simplest analogues of non-renormalisable quantum field theories (Klauder 1979, Kay 1981). Finally, making the substitution $t = -i\beta\hbar$, β being the inverse temperature, which transforms the propagator $K(x_2, x_1, t)$ to the equilibrium density matrix $\rho(x_2, x_1, \beta)$, we obtain in the quasi-classical approximation the generalisation of the classical Boltzmann distribution function for a particle in the uniform field taking into account the condition of the vanishing of the density matrix at a wall. It is worth noting that the question concerning such a generalisation was raised by Wang and Uhlenbeck as far back as 1945, but the complete explicit solution of this problem in the closed form has not been found until now.

2. The quasi-classical approximation for a propagator

Let us suppose that the propagator can be represented in the following 'collapsed' form

$$K(x_2, x_1, t) = \sum_n \alpha_n K^{(n)}(x_2, x_1, t) \tag{3}$$

$$K^{(n)}(x_2, x_1, t) = \exp[i\hbar^{-1} S^{(n)}(x_2, x_1, t) + i\chi(x_2, x_1, t) + i\hbar\varphi(x_2, x_1, t) + \ldots]$$

where the Planck constant is supposed to be 'small' and the constant coefficients α_n must be chosen in such a way that the conditions

$$K(0, x_1, t) = K(x_2, 0, t) \equiv 0 \tag{4}$$

would be fulfilled. (These conditions are the consequences of (2), because all the Schrödinger wavefunctions for the problem must vanish at a wall.) We demand each term of the expansion (3) separately to satisfy the Schrödinger equation for x_2, $x_1 > 0$, i.e.

$$i\hbar K_t = -\tfrac{1}{2}\hbar^2 m^{-1} K_{xx} - FxK = -\tfrac{1}{2}\hbar^2 m^{-1} K_{yy} - FyK. \tag{5}$$

(For convenience we drop the superscript (n) replacing the variables x_2, x_1 by x, y, respectively, also the notation $K_x \equiv \partial K/\partial x$, $K_{xy} \equiv \partial^2 K/\partial x\, \partial y$, is introduced.) Then for the function $S(x, y, t)$ we obtain the classical Hamilton-Jacobi equations

$$S_t + \tfrac{1}{2}m^{-1}S_x^2 - Fx = 0, \qquad S_t + \tfrac{1}{2}m^{-1}S_y^2 - Fy = 0. \tag{6}$$

The function $\chi(x, y, t)$ must satisfy the equations

$$\chi_t + m^{-1}\chi_x S_x - \tfrac{1}{2}\mathrm{i}m^{-1}S_{xx} = 0, \qquad \chi_t + m^{-1}\chi_y S_y - \tfrac{1}{2}\mathrm{i}m^{-1}S_{yy} = 0.$$

It is not difficult to verify that the solution of these equations is the following function

$$\chi(x, y, t) = -\tfrac{1}{2}\mathrm{i} \ln S_{xy} + \text{constant}. \tag{7}$$

Thus we obtain the well known Van Vleck formula (Van Vleck 1928, Berry and Mount 1972)

$$K(x, y, t) = \text{constant } S_{xy}^{1/2} \exp(\mathrm{i}\hbar^{-1}S(x, y, t) + \mathrm{i}\hbar\varphi(x, y, t) + \ldots). \tag{8}$$

The next term of the propagator phase expansion $\varphi(x, y, t)$ must satisfy the equations

$$\begin{aligned} \dot\varphi + \frac{1}{m}S_x\varphi_x + \frac{1}{8m}\left(\frac{S_{yxx}^2}{S_{xy}^2} - 2\frac{S_{yxxxx}}{S_{xy}}\right) = 0, \\ \dot\varphi + \frac{1}{m}S_y\varphi_y + \frac{1}{8m}\left(\frac{S_{xyy}^2}{S_{xy}^2} - 2\frac{S_{xyyy}}{S_{xy}}\right) = 0. \end{aligned} \tag{9}$$

It is essential that (7) and (9) are valid for quite arbitrary potentials. Consequently, if the action $S(x, y, t)$ is a quadratic form of coordinates, then formula (8) is exact. This situation takes place for systems with quadratic Hamiltonians in the absence of a wall. Indeed, for such systems solutions of the classical equations of motion are linear functions of the coordinates of the initial point x_1 and the final point x_2. Consequently, the action

$$S(x_2, t_2; x_1, t_1) = \int_{(x_1,t_1)}^{(x_2,t_2)} L(x(\tau), \dot x(\tau))\, \mathrm{d}\tau \tag{10}$$

(where L is Lagrange's function) quadratically depends on x_1 and x_2. (The explicit formula for the propagator of the Schrödinger equation with the Hamiltonian which is a multidimensional quadratic form of the coordinates and momenta operators with arbitrary time-dependent coefficients was given by Dodonov *et al* (1975).)

3. The free motion in a half space

Formula (8) allows us to obtain the exact result also for the problem of a free particle moving in the half space $x > 0$. In this case there are two classical trajectories connecting the initial and final points. The first one is the shortest trajectory (here after $t_1 = 0$)

$$x(\tau) = t^{-1}[x_1(t - \tau) + \tau x_2]$$

$$\dot x(\tau) = (x_2 - x_1)t^{-1}, \qquad 0 \leqslant \tau \leqslant t.$$

If we calculate its action by means of formula (10), and take into account the initial condition $K(x_2, x_1, t) \to \delta(x_2 - x_1)$ for $t \to 0$, we obtain the propagator of a free particle:

$$K_0(x_2, x_1, t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left(\frac{im(x_2 - x_1)^2}{2\hbar t}\right). \tag{11}$$

The second solution of classical equations corresponds to the motion with the absolutely elastic reflection from a wall at the moment of time $t_* = x_1 t/(x_1 + x_2)$:

$$x(\tau) = \begin{cases} t^{-1}[x_1(t-\tau) - x_2\tau], & 0 \leq \tau \leq t_* \\ t^{-1}[x_2\tau - x_1(t-\tau)], & t_* \leq \tau \leq t. \end{cases}$$

In the plane (t, x) this trajectory is obtained by means of connecting the points $(0, x_1)$ and $(t, -x_2)$ with the segment of a line and the following mirror reflection from the axis $x = 0$ of that part of the segment which lies in the region $x < 0$. Therefore formula (8) leads to the function which differs from K_0 only by the sign before the coordinate x_2:

$$K_0'(x_2, x_1, t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left(\frac{im(x_2 + x_1)^2}{2\hbar t}\right). \tag{12}$$

Taking into account conditions (4), we conclude that the propagator is the difference of functions K_0 and K_0':

$$K_0^{(w)}(x_2, x_1, t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \left[\exp\left(\frac{im(x_2 - x_1)^2}{2\hbar t}\right) - \exp\left(\frac{im(x_2 + x_1)^2}{2\hbar t}\right)\right]. \tag{13}$$

This result is exact. Note that the boundary condition $\psi(0) = 0$ is not the only possibility. For example, the exact propagator for the family of boundary conditions $\psi(0) = \beta\psi'(0)$, $-\infty < \beta < \infty$ were constructed by Clark *et al* (1980). We shall consider, however, the boundary condition $\psi(0) = 0$ only, bearing in mind the physical meaning of the problem under study as the problem of the motion of a particle in the half space bounded by the infinite high potential wall, from which a particle is reflected in an absolutely elastic manner. The condition $\psi(0) = 0$ appears naturally if equation (5) is considered as the equation for the radial part of the S-state wavefunction of the particle moving under the action of the spherically symmetric potential $V(r) = F|r|$ (this potential is known to have some relation to the problem of two quarks system).

The substitution $t = -i\beta\hbar$, where $\beta = (kT)^{-1}$, T absolute temperature, k Boltzmann's constant, reduces the propagator to the equilibrium density matrix

$$\rho_0^{(w)}(x_2, x_1, \beta) = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{1/2} \left[\exp\left(-\frac{m(x_2 - x_1)^2}{2\beta\hbar^2}\right) - \exp\left(-\frac{m(x_2 + x_1)^2}{2\beta\hbar^2}\right)\right]. \tag{14}$$

This formula was obtained by another method, for example, in Brown *et al* (1974). Besides, formula (13) is the limit case $g \to 0$, $w \to 0$ of a more general exact formula for the propagator of a time-dependent 'singular oscillator', i.e. the quantum system with the Hamiltonian $H = p^2/2m + mw^2x^2/2 + g/x^2$ which was considered in detail by Dodonov *et al* (1974) (see also Klauder 1979). Note that the propagator (13) has the form $K_0^{(w)}(x_2, x_1, t) = K_0(x_2, x_1, t) - K_0(x_2, -x_1, t)$, where K_0 is the reflectionless free propagator (11). A similar formula is valid for the propagator of the harmonic oscillator moving in the region $x > 0$. If we knew the full space propagator $\tilde{K}(x_2, x_1, t)$ corresponding to the potential $V(x) = -F|x|$, $-\infty < x < \infty$, then the propagator for the forced motion in the presence of the wall at $x = 0$ would be given by the analogous expression

$K_F^{(w)}(x_2, x_1, t) = \tilde{K}(x_2, x_1, t) - \tilde{K}(x_2, -x_1, t)$ (this remark was made by a referee). Unfortunately, the explicit form of $\tilde{K}$ is unknown, so that we are to deal with approximate methods to obtain the explicit form of $K_F^{(w)}$.

For completeness let us write the explicit formula for the equilibrium Wigner function corresponding to the density matrix (14):

$$\begin{aligned} W(q, p, \beta) &= \int_{-2q}^{2q} \rho(q + \tfrac{1}{2}\xi, q - \tfrac{1}{2}\xi, \beta) \exp(-i\hbar^{-1} p\xi)\, d\xi \\ &= \exp\left(-\frac{\beta p^2}{2m}\right) \operatorname{Re} \operatorname{erf}\left[q\left(\frac{2m}{\beta\hbar^2}\right)^{1/2} + ip\left(\frac{\beta}{2m}\right)^{1/2}\right] \\ &\quad - \left(\frac{2m}{\pi\beta p^2}\right)^{1/2} \sin\left(\frac{pq}{\hbar}\right) \exp\left(-\frac{2mq^2}{\beta\hbar^2}\right) \end{aligned} \tag{15}$$

where the error function erf(z) is defined as

$$\operatorname{erf}(z) = \frac{2}{\pi^{1/2}} \int_0^z \exp(-t^2)\, dt. \tag{16}$$

The function (15) tends to the classical distribution function $\exp(-\beta p^2/2m)$ when $mq^2/\beta\hbar^2 \gg 1$ or $\beta p^2/m \gg 1$. In the case of the motion in the full space in the presence of a force the solutions of the Schrödinger equation in the Wigner-Weyl representation were considered by Balazs (1980) for F = constant and by Dodonov *et al* (1984) for an arbitrary function $F(t)$.

4. The weak force case

Since in the free motion case formula (8) yields the exact result even in the presence of a wall, it seems natural to try to use the quasi-classical approach in the case of $F \neq 0$ as well. In the absence of the wall this approach leads to the exact formula (1) for the propagator (the quasi-classical solutions of the Schrödinger equation $H\psi = E\psi$ in this case were studied, e.g., by Crowley (1980)). Constants b, b' and τ_* are obtained from conditions of the absolute elasticity of a blow. As a result, for the parameter τ_* one arrives at the cubic equation

$$Fm^{-1}\tau_*^3 - \tfrac{3}{2}Fm^{-1}t\tau_*^2 - \tau_*(x_1 + x_2 - \tfrac{1}{2}Ft^2m^{-1}) + x_1 t = 0. \tag{17}$$

To understand qualitatively the behaviour of its solutions let us consider the case of $x_1 = x_2 = x$. Then one solution is obvious: $\tau_*^{(1)} = \frac{1}{2}t$, and it is not difficult to find two others:

$$\tau_*^{(2,3)} = \tfrac{1}{2}[t \pm (t^2 + 8mx/F)^{1/2}]. \tag{18}$$

If $F > 0$ (repulsive force), then both solutions are outside the interval $(0, t)$, so that in this case there exists a unique physically acceptable solution. If $F < 0$ (attractive force, i.e. the motion is finite), then the situation is rather complicated, because three solutions are possible, i.e. there exist three different classical trajectories with a blow. Really, in the case of $F < 0$ there are trajectories with any number of blows on a wall, i.e. the quantity of classical trajectories is countable. Therefore one could expect the quasi-classical expansion of the Green function has the form of (3), where every function K^n corresponds according to (8) to a certain classical trajectory. The situation is simplified if the quantity F is small. If the condition $|F|t^2/mx \ll 1$ is satisfied, then for

$F<0$ both solutions (18) become complex, so that the trajectory with one blow is unique. Under the same conditions trajectories with two and more blows are impossible. Indeed, if during the time t exactly N blows on a wall take place then the moment τ_* of the first blow can be found from the equation that is obtained in the same manner as equation (17):

$$N(N+1)F^2m^{-2}\tau_*^4-(2N+1)F^2m^{-2}t\tau_*^3-[(2N^2-1)2Fm^{-1}x_1+2Fm^{-1}x_2-F^2m^{-2}t^2]\tau_*^2 \\ +2Fm^{-1}(2N-1)x_1t\tau_*+4N(N-1)x_1^2=0. \quad (19)$$

For $N=1$ this equation is reduced to (17). If $x_1=x_2=x$ then equation (19) can be solved exactly. Four solutions are as follows

$$\tau_*^{(1,2)}=\frac{t}{2(N+1)}\pm\left(\frac{t^2}{4(N+1)^2}+\frac{2(N-1)xm}{(N+1)F}\right)^{1/2} \\ \tau_*^{(3,4)}=\frac{t}{2N}\pm\left(\frac{t^2}{4N^2}+\frac{2mx}{F}\right)^{1/2}. \quad (20)$$

As we see, if $F<0$ then all solutions become complex for $F\to 0$. If $F>0$ then for any F either τ_* or the time of final blow are outside the interval $(0, t)$. Thus, one can suppose that for $F\to 0$ only two terms remain in the series (3). The first of them is given by formula (1), the second must tend to expression (12) in the limit of $F=0$. Evidently, the time of the blow τ_* for $F\to 0$ will be slightly different from the time of a blow in a free case, i.e. the solutions of equation (17) can be found in the form of a series with respect to the parameter F. The first terms of the expansions of τ_* and $S^{(1)}$ are as follows

$$\tau_*=\frac{x_1t}{x_1+x_2}+\frac{x_1x_2(x_1-x_2)t^3}{2m(x_1+x_2)^4}F+\frac{x_1x_2[(x_1-x_2)^2-2x_1x_2]t^5}{(2m)^2(x_1+x_2)^7}F^2+\dots \quad (21)$$

$$S^{(1)}=\frac{m(x_1+x_2)^2}{2t}+\frac{Ft(x_1^2+x_2^2)}{2(x_1+x_2)}+\frac{F^2t^3}{2m}\left(\frac{x_1^2x_2^2}{(x_1+x_2)^4}-\frac{1}{12}\right)+\dots. \quad (22)$$

The following terms of action's expansion in powers of F can be found with the aid of equations (6). From formula (22) one can make the conclusion that the function $S^{(1)}$ has the following functional form:

$$S^{(1)}=\frac{m(x_1+x_2)^2}{2t}[1+\psi(\mu, z)], \qquad \mu=\frac{Ft^2}{m(x_1+x_2)}, \qquad z=\left(\frac{x_2-x_1}{x_1+x_2}\right)^2.$$

Taking the sum of equations (6) and introducing new variables we obtain the equation for the function $\psi(\mu, z)$

$$\psi+\mu\frac{\partial\psi}{\partial\mu}+\frac{1}{4}\left(2\psi-\mu\frac{\partial\psi}{\partial\mu}\right)^2+z(1-z)\left(\frac{\partial\psi}{\partial z}\right)^2-\mu(1+z)=0. \quad (23)$$

Further, we present the function ψ in the form

$$\psi(\mu, z)=\sum_{k=1}^{\infty}\mu^k\psi_k(z).$$

For the function $\psi_1(z)$ one can find the expression

$$\psi_1(z)=\tfrac{1}{2}(1+z) \quad (24)$$

which conforms with formula (22). Other functions are calculated from the recurrent formula

$$\psi_k(z)=[4(k+1)]^{-1}\sum_{j=1}^{k-1}[4z(z-1)\psi_j'(z)\psi_{k-j}'(z)-(j-2)(k-j-2)\psi_j(z)\psi_{k-j}(z)]. \quad (25)$$

In particular

$$\psi_2(z)=\tfrac{1}{16}(z-1)^2-\tfrac{1}{12},\qquad \psi_3(z)=\tfrac{1}{32}z(z-1)^2,\qquad \psi_4(z)=\tfrac{1}{128}z(z-1)^2(3z-1). \quad (26)$$

It follows from formulae (24) and (25) that for $k\geqslant 3$ the functions $\psi_k(z)$ have the form

$$\psi_k(z)=z(z-1)^2p_k(z) \quad (27)$$

where the functions $p_k(z)$ are polynomials of the degree $k-3$. One can obtain the following formula for the cross derivative:

$$\frac{\partial^2 S^{(1)}}{\partial x_1\,\partial x_2}=\frac{m}{t}\left[1+\sum_{k=1}^{\infty}\mu^k\left(\tfrac{1}{2}(k-2)(k-1)\psi_k(z)\right.\right.$$
$$\left.\left.+[(2k-1)z-1]\psi_k'(z)+2z(z-1)\psi_k''(z)\right)\right]. \quad (28)$$

Due to formulae (24)-(27) this derivative can be written in the form

$$\frac{\partial^2 S^{(1)}}{\partial x_1\,\partial x_2}=\frac{m}{t}\left(1+(z-1)\sum_{k=1}^{\infty}\mu^k\tilde{p}_k(z)\right) \quad (29)$$

where $\tilde{p}_k(z)$ are polynomials of degree $k-1$. At a wall (for $x_1=0$ or $x_2=0$) one has $z=1$, therefore both the action $S^{(1)}$ and the modulus of its cross derivative are the same as for the action S_f. Therefore, we find the following expression for the propagator

$$K(x_2,x_1,t)=\left(\frac{m}{2\pi i\hbar t}\right)^{1/2}\left[\!\!\left[\exp\left[\frac{i}{\hbar}\left(\frac{m(x_2-x_1)^2}{2t}+\frac{Ft(x_1+x_2)}{2}-\frac{F^2t^3}{24m}\right)\right]\right.\right.$$
$$-\left[1+(z-1)\left(\tfrac{1}{2}\mu+\tfrac{1}{8}\mu^2(5z-1)+\tfrac{1}{32}\mu^3(28z^2-17z+1)+\sum_{k=4}^{\infty}\mu^4\tilde{p}_k(z)\right)\right]^{1/2}$$
$$\times\exp\left\{\frac{i}{\hbar}\left[\frac{m(x_1+x_2)^2}{2t}+\frac{Ft(x_1^2+x_2^2)}{2(x_1+x_2)}+\frac{F^2t^3}{24m}\left(12\frac{x_1^2x_2^2}{(x_1+x_2)^4}-1\right)\right.\right.$$
$$\left.\left.\left.\left.+\frac{8mx_1^2x_2^2(x_2-x_1)^2}{t(x_1+x_2)^4}\left(\tfrac{1}{32}\mu^3+\tfrac{1}{128}\mu^4(3z-1)+\sum_{k=5}^{\infty}\mu^kp_k(z)\right)\right]\right\}\right]\!\!\right]. \quad (30)$$

As to the quantum correction $\varphi(x_2,x_1,t)$ (see formula (3)), it follows from equation (9) that it can be presented in the form of

$$\varphi=\frac{t}{m(x_1+x_2)^2}\sum_{k=1}^{\infty}\mu^k\gamma_k(z).$$

(The expansion must begin with $k=1$, because formula (30) coincides with formula (13) for $F=0=\mu$.) The first two terms of the expansion are as follows

$$\varphi=\frac{t}{m(x_1+x_2)^2}\left(-\tfrac{3}{4}\mu(z-1)-\tfrac{1}{32}\mu^2(z-1)(69z+11)+\ldots\right).$$

(It is essential that constants appearing in integrating linear non-uniform equations

(9) can always be chosen in such a way that the correction $\varphi(\mu, z)$ would be equal to zero at a wall, i.e. for $z=1$.) Hence formula (30) is true under the conditions

$$\frac{\hbar F t^3 x_1 x_2}{m^2(x_1+x_2)^5} \ll 1; \qquad \frac{Ft^2}{m(x_2+x_1)} \ll 1. \tag{31}$$

For the equilibrium density matrix one obtains the expression (we write $F=-f, f>0$ because in the case discussed only the attraction field should be considered):

$$\rho(x_2, x_1, \beta) = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{1/2} \left\{ \exp\left(-\frac{m(x_2-x_1)^2}{2\beta\hbar^2} - \frac{f\beta}{2}(x_1+x_2) + \frac{f^2\beta^3\hbar^2}{24m} \right) \right.$$
$$- \left(1 + (z-1) \sum_{k=1}^{\infty} \tilde{\mu}^k \tilde{p}_k(z)\right)^{1/2} \exp\left[-\frac{m(x_2+x_1)^2}{2\beta\hbar^2} - \frac{f\beta(x_1^2+x_2^2)}{2(x_1+x_2)} \right.$$
$$- \frac{f^2\beta^3\hbar^2}{24m}\left(12\frac{x_1^2x_2^2}{(x_1+x_2)^4} - 1\right)$$
$$\left.\left. - \frac{8mx_1^2x_2^2(x_2-x_1)^2}{\beta\hbar^2(x_2+x_1)^4} \sum_{k=3}^{\infty} \tilde{\mu}^k p_k(z) \right] \right\}, \qquad \tilde{\mu} = \frac{f\beta^2\hbar^2}{m(x_1+x_2)}. \tag{32}$$

Let us pay attention to the fact that for $x_1 = x_2 = x$ one obtains the closed expression for the function $S^{(1)}$

$$S^{(1)}(x, x, t) = 2mx^2/t + \tfrac{1}{2}Ftx - F^2t^3/96m \tag{33}$$

(since we know the exact solution of (17) $\tau_* = \frac{1}{2}t$). In this case it is also possible to find a simple expression for the cross derivative. Introducing the variables $x = \frac{1}{2}(x_1+x_2)$ and $\eta = x_1 - x_2$ we shall look for $S^{(1)}$ in the form of

$$S^{(1)}(x, \eta, t) = g_0(x, t) + \sum_{k=1}^{\infty} \eta^k g_k(x, t)$$

where the function $g_0(x, t)$ is given by formula (33). In order to find the functions $g_k(x, t)$ we write the difference of equations (6) in the new variables:

$$m^{-1}(\partial S/\partial x)\, \partial S/\partial\eta - F\eta = 0. \tag{34}$$

From this equation we obtain

$$g_1 = 0, \qquad g_2 = mFt/(Ft^2 + 8mx).$$

Thus

$$\left.\frac{\partial^2 S^{(1)}}{\partial x_1\, \partial x_2}\right|_{x_1=x_2} = \left.\left(\frac{1}{4}\frac{\partial^2 S^{(1)}}{\partial x^2} - \frac{\partial^2 S^{(1)}}{\partial \eta^2}\right)\right|_{\eta=0} = \frac{m(8mx - Ft^2)}{t(8mx + Ft^2)}.$$

Consequently, we obtain the following generalisation of the classical Boltzmann formula for the diagonal elements of the equilibrium density matrix

$$\rho(x, x, \beta) = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{1/2} \left[\exp\left(-f\beta x + \frac{f^2\beta^3\hbar^2}{24m} \right) - \left(\frac{8mx - f\beta^2\hbar^2}{8mx + f\beta^2\hbar^2} \right)^{1/2} \right.$$
$$\left. \times \exp\left(-\frac{2mx^2}{\beta\hbar^2} - \tfrac{1}{2}f\beta x + \frac{f^2\beta^3\hbar^2}{96m} \right) \right]. \tag{35}$$

This formula is true under the following conditions

$$f\beta^2\hbar^2/mx \ll 1, \qquad f\beta^3\hbar^4/m^2x^3 \ll 1, \tag{36}$$

i.e. for relatively weak fields, high temperatures and far from the wall. In practice, formula (35) can be interesting only provided it is true at least for the value of the coordinate of the order of de Broglie's wavelength $x_* = (\beta\hbar^2/m)^{1/2}$ when the second exponential in (35) is not too small. Introducing $x = x_*$ to the inequality (36) we obtain the following restriction on the strength of field

$$\phi = f^2\hbar^2/m(kT)^3 \ll 1. \tag{37}$$

In a gravitational field condition (37) is always fulfilled: for an electron in the Earth's gravitational field we have $\phi \sim 3\times10^{-28}$ for $T = 1$ K. (Note, that the term $f\beta x$ is also very small for all reasonable values of β and x in this case, i.e. the density matrix coincides practically with (14)). The strength of the electrical field under the same conditions should be sufficiently smaller than 300 V/m. Formulae (35) and (32) can be simplified if the condition (37) is fulfilled as follows

$$\rho(x_2, x_1, \beta) \approx \left(\frac{m}{2\pi\beta\hbar^2}\right)^{1/2}\left\{\exp\left[-\frac{m(x_2-x_1)^2}{2\beta\hbar^2} - \frac{f\beta(x_1+x_2)}{2}\right]\right.$$

$$\left. - \exp\left(-\frac{m(x_1+x_2)^2}{2\beta\hbar^2} - \frac{f\beta(x_1^2+x_2^2)}{2(x_2+x_1)}\right)\right\}, \qquad f\beta^2\hbar^2\frac{x_1x_2}{(x_1+x_2)^3} \ll 1 \tag{38}$$

$$\rho(x, x, \beta) \approx \left(\frac{m}{2\pi\beta\hbar^2}\right)^{1/2}\left[\exp(-f\beta x) - \exp\left(-\frac{2mx^2}{\beta\hbar^2} - \tfrac{1}{2}f\beta x\right)\right], \qquad \frac{f\beta^2\hbar^2}{mx} \ll 1. \tag{39}$$

5. The strong field case

One can also obtain the approximate solutions of (19) and (6) in the form of series of x_1 and x_2. Confining ourselves to the terms of order y_k^2 with respect to the dimensionless variables $y_k = mx_k/Ft^2$ $(k = 1, 2)$ (this means that we suppose that $|y| \ll 1$, i.e. the field is strong) one can obtain four sets of solutions to (19):

$$\tau_A^{(N)} = -2(N-1)ty_1[1 - 2N(N-1)y_1 - 2(N-1)y_2 + 8N^2(N-1)^2y_1^2$$

$$+ 4(N-1)^2(3N+1)y_1y_2 + 4(N-1)(N^2-1)y_2^2 + \ldots] \tag{40}$$

$$\tau_B^{(N)} = -2Nty_1[1 - 2N(N+1)y_1 + 2Ny_2 + 8N^2(N+1)^2y_1^2$$

$$- 4N^2(3N+4)y_1y_2 - 4N^2(N-2)y_2^2 + \ldots] \tag{41}$$

$$\tau_C^{(N)} = (t/N)[1 + 2N(N-1)y_1 + 2Ny_2 - 4N^2(N-1)^2y_1^2 - 4N^2(N-2)y_1y_2$$

$$- 4N^2(N+1)y_2^2 + \ldots] \tag{42}$$

$$\tau_D^{(N)} = [t/(N+1)][1 + 2N(N+1)y_1 - 2(N+1)y_2 - 4N^2(N+1)^2y_1^2$$

$$+ 4(N+1)(N^2-1)y_1y_2 + 4N(N+1)^2y_2^2 + \ldots]. \tag{43}$$

These formulae determine the moment of the first blow. The moment of the nth blow is then equal to

$$\tau_n = n\tau_* - 2(n-1)mx_1/F\tau_*, \qquad n = 1, 2, \ldots, N. \tag{44}$$

The action in the case of N blows equals

$$S^{(N)}=\frac{m}{2}\left(\frac{x_1^2}{\tau_*}+\frac{x_2^2}{t-\tau_N}\right)+\frac{F}{2}[x_1\tau_*+x_2(t-\tau_*)]$$
$$-\frac{F^2}{24m}[\tau_*^3+(N-1)(\tau_n-\tau_{n-1})^3+(t-\tau_N)^3]. \quad (45)$$

Thus one gets four sets of action functions

$$S_A^{(N)}=\frac{F^2t^3}{m}\left(-\frac{1}{24(N-1)^2}-\frac{y_1+y_2}{2(N-1)}-\frac{2N-1}{2}(y_1^2+y_2^2)-y_1y_2+\ldots\right), \qquad N\geqslant 2 \quad (46)$$

$$S_B^{(N)}=\frac{F^2t^3}{m}\left(-\frac{1}{24N^2}+\frac{y_2-y_1}{2N}-\frac{2N+1}{2}y_1^2+\frac{2N-1}{2}y_2^2+y_1y_2+\ldots\right), \qquad N\geqslant 1 \quad (47)$$

$$S_C^{(N)}=\frac{F^2t^3}{m}\left(-\frac{1}{24N^2}+\frac{y_1-y_2}{2N}+\frac{2N-1}{2}y_1^2-\frac{2N+1}{2}y_2^2+y_1y_2+\ldots\right), \qquad N\geqslant 1 \quad (48)$$

$$S_D^{(N)}=\frac{F^2t^3}{m}\left(-\frac{1}{24(N+1)^2}+\frac{y_1+y_2}{2(N+1)}+\frac{2N+1}{2}(y_1^2+y_2^2)-y_1y_2+\ldots\right). \qquad N\geqslant 0. \quad (49)$$

For all these functions $|\partial^2S/\partial x_1\,\partial x_2|=m/t$. Moreover, $S_D^{(0)}$ coincides with the reflectionless action S_f. The condition (4) is satisfied provided one writes

$$K=\left(\frac{m}{2\pi i\hbar t}\right)^{1/2}\sum_{N=0}^{\infty}(-1)^N\left[\exp\left(\frac{i}{\hbar}S_D^{(N)}\right)-\exp\left(\frac{i}{\hbar}S_C^{(N+1)}\right)\right.$$
$$\left.-\exp\left(\frac{i}{\hbar}S_B^{(N+1)}\right)+\exp\left(\frac{i}{\hbar}S_A^{(N+2)}\right)\right]. \quad (50)$$

Formally, we have obtained the quasi-classical propagator in the strong field case. However, expression (50) is quite unsatisfactory from the physical point of view. Indeed, all the terms in this expression have the leading factors $\exp(-iF^2t^3/24mN^2)$ (since $|y_k|\ll 1$). Consequently, if one writes the equilibrium density matrix, then the leading terms in the exponentials will be proportional to the expression $f^2\beta^3\hbar^2/m$, which can be written up to a number coefficient as $(E_0/kT)^3$, E_0 being the energy of the ground state (Flügge 1971). But we know that the temperature can enter the equilibrium density matrix only in the form of $\exp(-E_0/kT)$, if $T\to 0$.

This means that the propagator does not 'collapse' in Crandall's sense in the strong field case. In other words, in the strong field case the quasi-classical approach fails.

Acknowledgments

The authors thank the referees for useful remarks and suggestions which helped to improve the first version of the paper.

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